Illumination freeform design using Monge-Ampère equations

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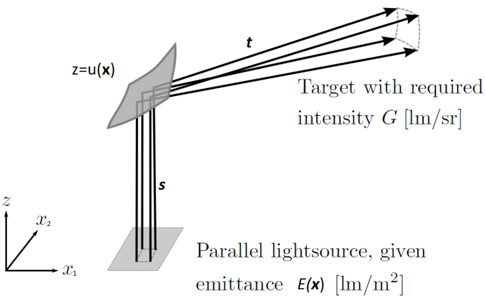


SINGLE REFLECTOR, PARALLEL IN, FAR FIELD OUT

- reflector surface $\mathcal{R}$ $\quad z = u(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{S}$ (source)
- physical principles
  1. law of reflection
  2. conservation of energy
- assumptions
  1. perfect specular reflection
  2. far field approximation
- transport plan
  1. emittance E $[\mathrm{lm/m^2}]$ $\quad \rightarrow \quad$ intensity G $[\mathrm{lm/sr}]$
  2. space coordinates $\boldsymbol{x}$ $\quad \rightarrow \quad$ angular coordinates $\boldsymbol{\psi} = (\psi, \chi)$
- spherical coordinates

  zenith ψ $(0 \leq \psi \leq \pi)$, azimuth χ $(0 \leq \chi < 2\pi)$

SINGLE REFLECTOR, PARALLEL IN, FAR FIELD OUT

- target domain $\mathcal{T}$
- two-stage approach
  1. determine optical map $\boldsymbol{m} : \mathcal{S} \to \mathcal{T}$
  2. determine location/shape reflector surface

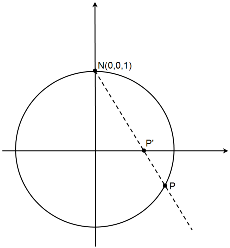
Single reflector, parallel in, far field out

- law of reflection $\quad \boldsymbol{t} = \boldsymbol{s} - 2(\boldsymbol{s} \cdot \boldsymbol{\nu})\boldsymbol{\nu}, \quad \boldsymbol{s} = \boldsymbol{e}_z$
- in terms of gradient: $\boldsymbol{t} = \boldsymbol{t}(\nabla u(\boldsymbol{x}))$
- alternative: stereographic coordinates

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{1 - t_3} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

- optical map $\boldsymbol{m} : \mathcal{S} \to \mathcal{T}$

$$\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x}) = \nabla u(\boldsymbol{x})$$



Single reflector, parallel in, far field out

- energy conservation ($\mathcal{A} \subset \mathcal{S}$)

$$\int_{\mathcal{A}} E(\boldsymbol{x})\,\mathrm{d}A(\boldsymbol{x}) = \int_{t(\mathcal{A})} G(\psi(\boldsymbol{t}))\,\mathrm{d}S(\boldsymbol{t})$$

- coordinate transformations target: $\psi \to \boldsymbol{t} \to \boldsymbol{y} \in \mathcal{T}$
- assumption: $\det\big(\mathrm{D}\boldsymbol{m}\big) > 0$

$\boxed{\boldsymbol{y} = \nabla u}$

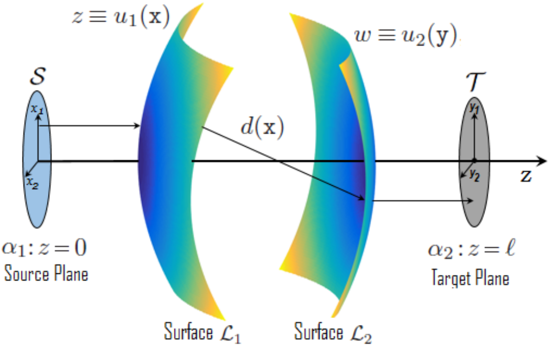
- energy balance, Monge-Ampère equation

$$E(\boldsymbol{x}) = G(\psi(\boldsymbol{t}))\, \underbrace{\frac{4}{\left(|\boldsymbol{y}|^2 + 1\right)^2}}_{\text{map } \boldsymbol{y} \longrightarrow \boldsymbol{t}}\, \underbrace{\det\left(\mathrm{D}^2 u(\boldsymbol{x})\right)}_{\text{map } \boldsymbol{x} \longrightarrow \boldsymbol{y}}$$

- BVP Monge-Ampère equation

$$\begin{aligned} \det\left(\mathrm{D}^2 u(\boldsymbol{x})\right) &= F(\boldsymbol{x}, \nabla u), \quad \boldsymbol{x} \in \mathcal{S} \\ \nabla u(\partial\mathcal{S}) &= \partial\mathcal{T} \qquad \text{(transport BC)} \end{aligned}$$

Two lens surfaces, parallel in, parallel out



TWO LENS SURFACES, PARALLEL IN, PARALLEL OUT

- lens surfaces

$$\mathcal{L}_1 : z = u_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{S}(\text{source})$$
$$\mathcal{L}_2 : \ell - z = u_2(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathcal{T}(\text{target})$$

- physical principles
  1. Snell's law of refraction
  2. conservation of energy
- assumption: no Fresnel reflection
- transport plan
  1. emittance E $[\mathrm{lm}/\mathrm{m}^2]$ $\rightarrow$ illuminance G $[\mathrm{lm}/\mathrm{m}^2]$
  2. space coordinates $\boldsymbol{x}$ $\rightarrow$ space coordinates $\boldsymbol{y}$

Two lens surfaces, parallel in, parallel out

- two-stage approach
  1. determine optical map $\boldsymbol{m} : \mathcal{S} \to \mathcal{T}$
  2. determine location lens surfaces

TWO LENS SURFACES, PARALLEL IN, PARALLEL OUT

- law of refraction $\boldsymbol{t} = \eta \boldsymbol{s} + F\big(\nabla u_1; \eta\big)\boldsymbol{\nu}_1, \quad \boldsymbol{s} = \boldsymbol{e}_z$
- principle of equal optical paths

$$L = u_1(\boldsymbol{x}) + nd(\boldsymbol{x}) + u_2(\boldsymbol{y}) = \text{Const.}$$

- optical map $\boldsymbol{m} : \mathcal{S} \to \mathcal{T}$

$$\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{x} - \frac{\beta \nabla u_1(\boldsymbol{x})}{\sqrt{n^2 + (n^2 - 1)|\nabla u_1|^2}}$$

- location of lens surfaces, cost function

$$u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = c(\boldsymbol{x}, \boldsymbol{y}), \quad c(\boldsymbol{x}, \boldsymbol{y}) = a_1 - \sqrt{a_2 - a_3|\boldsymbol{x} - \boldsymbol{y}|^2}$$

- energy balance ($\mathcal{A} \subset \mathcal{S}$)

$$\int_{\mathcal{A}} E(\boldsymbol{x})\mathrm{d}A(\boldsymbol{x}) = \int_{\boldsymbol{m}(\mathcal{A})} G(\boldsymbol{y})\mathrm{d}A(\boldsymbol{y})$$

- assumption: $\det\big(\mathrm{D}\boldsymbol{m}\big) > 0$
- BVP generalized Monge-Ampère equation

$$\det\big(\mathrm{D}\boldsymbol{m}(\boldsymbol{x})\big) = \frac{E(\boldsymbol{x})}{G(\boldsymbol{m}(\boldsymbol{x}))}, \quad \boldsymbol{x} \in \mathcal{S}$$
$$\boldsymbol{m}(\partial\mathcal{S}) = \partial\mathcal{T} \qquad \text{(transport BC)}$$

- possible solution ('maximum solution')

$$u_1(\boldsymbol{x}) = \max_{\boldsymbol{y}\in\mathcal{T}} \big(c(\boldsymbol{x},\boldsymbol{y}) - u_2(\boldsymbol{y})\big), \quad u_2(\boldsymbol{y}) = \max_{\boldsymbol{x}\in\mathcal{S}} \big(c(\boldsymbol{x},\boldsymbol{y}) - u_1(\boldsymbol{x})\big)$$

- necessary condition: $\nabla_{\boldsymbol{x}} c(\boldsymbol{x},\boldsymbol{y}) - \nabla u_1(\boldsymbol{x}) = \boldsymbol{0} \Leftrightarrow \boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x})$
- sufficient condition: $\mathrm{D}_{\boldsymbol{xx}} c(\boldsymbol{x},\boldsymbol{m}(\boldsymbol{x})) - \mathrm{D}^2 u_1(\boldsymbol{x})$ SND
- (matrix-)equation for $\boldsymbol{m}$

$$\boldsymbol{C}\mathrm{D}\boldsymbol{m}(\boldsymbol{x}) = \mathrm{D}^2 u_1(\boldsymbol{x}) - \mathrm{D}_{\boldsymbol{xx}} c = \boldsymbol{P}, \quad \boldsymbol{C} = \mathrm{D}_{\boldsymbol{xy}} c$$

- generalized Monge-Ampère equation

$$\det\big(\mathrm{D}\boldsymbol{m}(\boldsymbol{x})\big) = \frac{\det(\boldsymbol{P}(\boldsymbol{x}))}{\det(\boldsymbol{C}(\boldsymbol{x},\boldsymbol{m}(\boldsymbol{x})))} = \frac{E(\boldsymbol{x})}{G(\boldsymbol{m}(\boldsymbol{x}))}$$

- optical map, location/shape of optical surface(s)

$$\boldsymbol{y} = \boldsymbol{m}(\boldsymbol{x}), \quad u_1(\boldsymbol{x}) + u_2(\boldsymbol{y}) = c(\boldsymbol{x}, \boldsymbol{y})$$

REFLECTOR	LENS
$\boldsymbol{m} = \boldsymbol{x} - \nabla u_1 = \nabla u$	$\boldsymbol{m} = \boldsymbol{x} - \boldsymbol{f}(\nabla u_1)$
$c(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\lvert\boldsymbol{x} - \boldsymbol{y}\rvert^2$	$c(\boldsymbol{x}, \boldsymbol{y}) = f(\lvert\boldsymbol{x} - \boldsymbol{y}\rvert^2)$
$\boldsymbol{C} = \boldsymbol{I}$	$\boldsymbol{C} = \mathrm{D}_{\boldsymbol{xy}}c$

- energy conservation: $\det(\boldsymbol{C}\mathrm{D}\boldsymbol{m}) = F(\boldsymbol{x}, \boldsymbol{m})$

REFLECTOR	LENS
$\det\left(\mathrm{D}^2 u\right) = F(\boldsymbol{x}, \nabla u)$	$\det\left(\mathrm{D}^2 u_1 - \mathrm{D}_{\boldsymbol{xx}}c\right) = F(\boldsymbol{x}, \boldsymbol{m})$

- determine a mapping $\boldsymbol{m} : \mathcal{S} \to \mathcal{T}$, containing ∇u_1, such that

$$\det\left(\boldsymbol{C}\mathrm{D}\boldsymbol{m}(\boldsymbol{x})\right) = F(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathcal{S}$$
$$\boldsymbol{m}(\partial\mathcal{S}) = \partial\mathcal{T}$$

- to compute $u_1(\boldsymbol{x})$, minimize the functional

$$I[u] = \int_{\mathcal{S}} \|\nabla u_1 - \nabla_{\boldsymbol{x}} c\|^2 \,\mathrm{d}A$$

- minimize the following functionals:

$$
\begin{aligned}
&\text{interior} && J_{\mathrm{I}}[\boldsymbol{m},\boldsymbol{P}] = \tfrac{1}{2}\int_{\mathcal{S}} \|\mathcal{C}\mathrm{D}\boldsymbol{m} - \boldsymbol{P}\|^2 \,\mathrm{d}A \\
&\quad\text{subject to} && \det\big(\boldsymbol{P}(\boldsymbol{x})\big) = F(\boldsymbol{x},\boldsymbol{m}(\boldsymbol{x})), \quad \boldsymbol{P}(\boldsymbol{x})\ \text{SPD} \\
&\text{boundary} && J_{\mathrm{B}}[\boldsymbol{m},\boldsymbol{b}] = \tfrac{1}{2}\int_{\partial\mathcal{S}} \|\boldsymbol{m} - \boldsymbol{b}\|^2 \,\mathrm{d}s \\
&\text{combined} && J[\boldsymbol{m},\boldsymbol{P},\boldsymbol{b}] = \alpha J_{\mathrm{I}}(\boldsymbol{m},\boldsymbol{P}) + (1-\alpha) J_{\mathrm{B}}(\boldsymbol{m},\boldsymbol{b})
\end{aligned}
$$

- vector field on boundary, $\boldsymbol{b} : \partial\mathcal{S} \to \partial\mathcal{T}$

ITERATION SCHEME

- initial guess $\boldsymbol{m}^0$, $\mathrm{D}\boldsymbol{m}^0$ SPD
- iterate for $n = 0, 1, 2, \ldots$

$$\boldsymbol{b}^{n+1} = \mathrm{argmin}_{\boldsymbol{b}\in\mathcal{B}} J_{\mathrm{B}}[\boldsymbol{m}^n, \boldsymbol{b}]$$

$$\boldsymbol{P}^{n+1} = \mathrm{argmin}_{\boldsymbol{P}\in\mathcal{P}(\boldsymbol{m}^n)} J_{\mathrm{I}}[\boldsymbol{m}^n, \boldsymbol{P}]$$

$$\boldsymbol{m}^{n+1} = \mathrm{argmin}_{\boldsymbol{m}\in\mathcal{M}} J[\boldsymbol{m}, \boldsymbol{P}^{n+1}, \boldsymbol{b}^{n+1}]$$

COMPUTATION OF $\boldsymbol{P}$

$$\boxed{J_{\mathrm{I}}[\boldsymbol{m},\boldsymbol{P}] = \frac{1}{2}\int_{\mathcal{S}} \|\boldsymbol{C}\mathrm{D}\boldsymbol{m} - \boldsymbol{P}\|^2 \,\mathrm{d}A}$$

- difference approximation $\boldsymbol{D} \approx \mathrm{D}\boldsymbol{m}$
- constraint minimization problem

$$\begin{aligned} \text{Minimize} \quad & H(p_{11}, p_{22}, p_{12}) = \tfrac{1}{2}\|\boldsymbol{C}\boldsymbol{D} - \boldsymbol{P}\|^2 \\ \text{subject to} \quad & \det(\boldsymbol{P}) = p_{11}p_{22} - p_{12}^2 = F(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x})) \\ & \mathrm{tr}(\boldsymbol{P}) = p_{11} + p_{22} \geq 0 \end{aligned}$$

- nonlinear problem, exact solution available
- interpretation: conservation of energy is enforced locally
- $\mathrm{tr}(\boldsymbol{P}) \geq 0$ corresponds to convex lens surface $z = u_1(\boldsymbol{x})$

Minimization of J

- necessary/sufficient condition for minimum $J[\boldsymbol{m}, \boldsymbol{P}, \boldsymbol{b}]$

$$\delta J[\boldsymbol{m}, \boldsymbol{P}, \boldsymbol{b}](\boldsymbol{\eta}) = 0$$

- elliptic equation system for $\boldsymbol{m}$

$$\nabla \cdot \left(\boldsymbol{C}^{\mathrm{T}} \boldsymbol{C} \mathrm{D} \boldsymbol{m}\right) = \nabla \cdot \left(\boldsymbol{C}^{\mathrm{T}} \boldsymbol{P}\right) \quad \boldsymbol{x} \in \mathcal{S} \quad + \quad \mathrm{BC}$$

- necessary/suffient condition for minimum $I[u_1]$

$$\delta I[u_1](v) = 0$$

- BVP for first reflector surface

$$\nabla^2 u_1 = \nabla \cdot \nabla_{\boldsymbol{x}} c(\boldsymbol{x}, \boldsymbol{m}) \quad \boldsymbol{x} \in \mathcal{S} \quad + \quad \mathrm{BC}$$

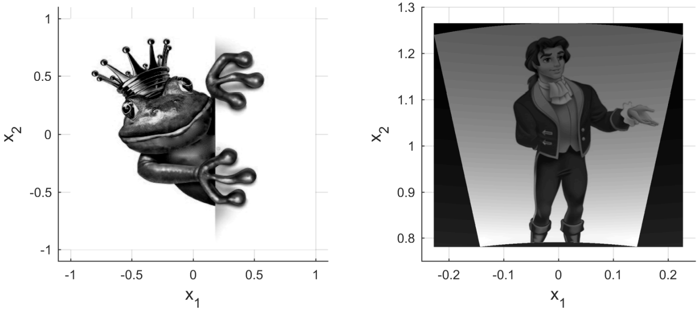
- second lens surface

$$u_2(\boldsymbol{m}(\boldsymbol{x})) = c(\boldsymbol{x}, \boldsymbol{m}(\boldsymbol{x})) - u_1(\boldsymbol{x}) \quad \boldsymbol{x} \in \mathcal{S}$$

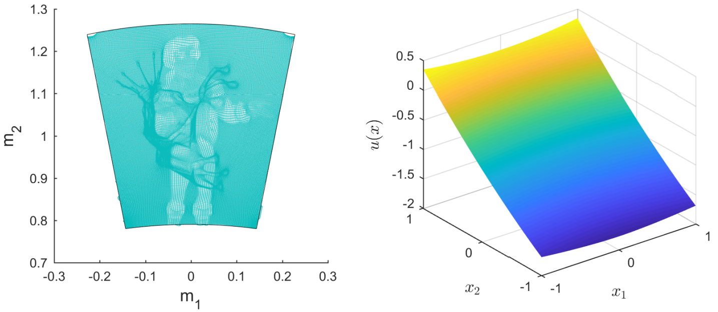
PROPERTIES OF ALGORITHM

- two-stage method: mapping + optical surface
- least-squares approximation + iteration for mapping
- local nonlinear problems: BC, energy conservation
- linear elliptic PDE for mapping
- Poisson equation for optical surface

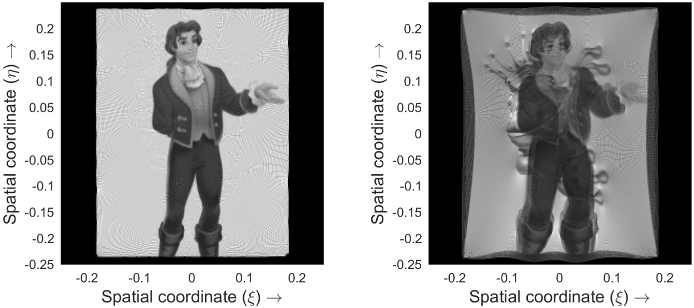
Freeform reflector: from frog to prince



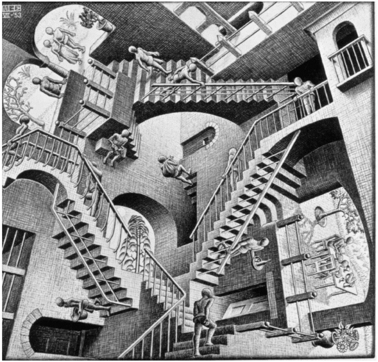
From frog to prince: mapping and reflector surface



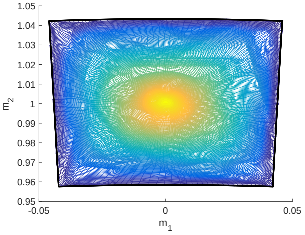
From frog to prince: ray trace result

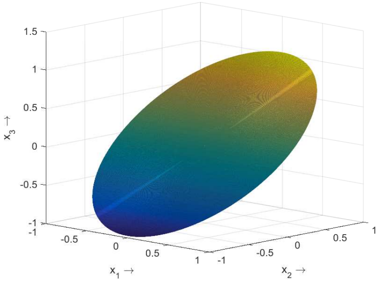


ESCHER LITHOGRAPH

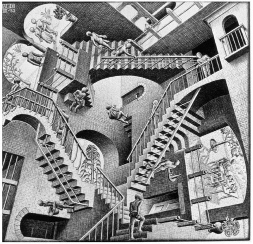


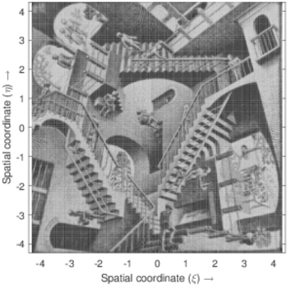
ESCHER LITOGRAPH: MAPPING AND REFLECTOR SURFACE



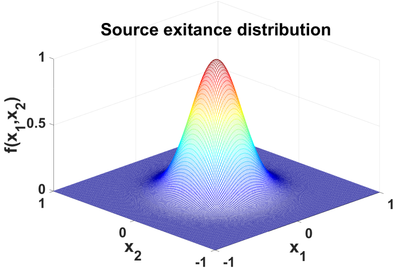


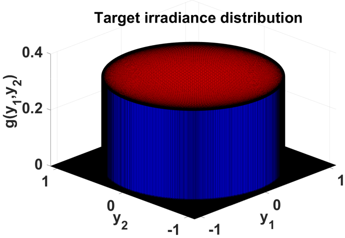
Escher litograph: original and ray trace image



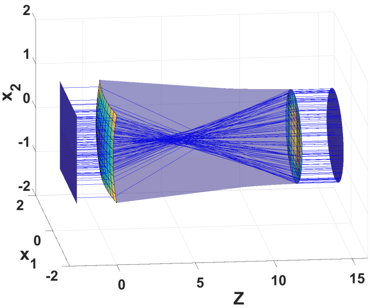


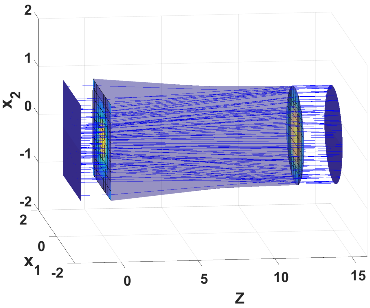
Laser beam shaping: from Gaussian to circular top-hat



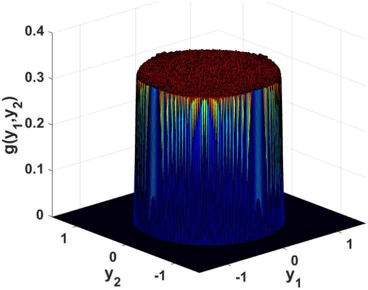


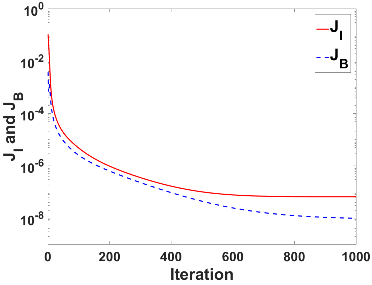
SINGLE LENS: 'MAXIMUM/MINIMUM' SOLUTIONS



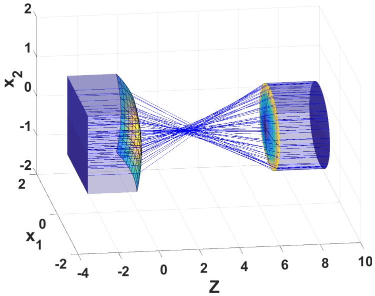


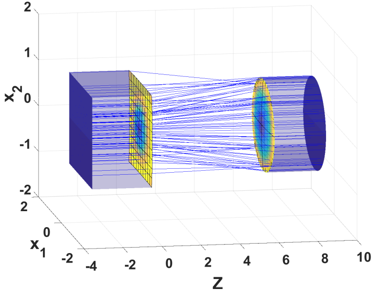
SINGLE LENS: TARGET DISTRIBUTION AND RESIDUALS





Two lenses: target 'maximum/minimum' solutions





Concluding remarks

- Monge-Ampère BVP to compute free-form surfaces
- based on optical map and energy conservation
- algorithm for inverse problem: mapping + optical surface
- least-squares approximation + iterative scheme for mapping
- Poisson equation for reflector, coupled elliptic PDE for lens
- very efficient algorithm
- (very) complicated source/target distributions possible, high contrast
- numerical tool for optical designs

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